POSITIVITY OF THE EXPONENT FOR STATIONARY SEQUENCES OF MATRICES.

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Consider a stationary sequence of $(d \times d)$ real matrices $\{A_n, n \in Z\}$ and the products $\{A^{(n)} = A_{n-1} \dots A_o, n \geqq 0\}$. With natural integrability conditions, the sequences of numbers $\{E(\frac{1}{n} \log||A^{(n)})||, n \geqq 0\}$ and $\{-E(\frac{1}{n} \log||(A^{(n)})^{-1}||), n \geqq 0\}$ converge towards numbers γ_+ and γ_-. Clearly $\gamma_+ \geqq \gamma_-$ and a typical feature of such a product is that inequality in general holds. We are interested here in describing necessary conditions under which equality holds.

In the independent case criterions go back to Furstenberg [F]. They were generalized to the Markov case by Virtser [V], Guivarc'h [G] and Royer [R]. Weaker independence conditions are considered in [LR]. In a different framework, Kotani's theory (see [S]) deals with Jacobi matrices depending on a parameter E and yields that if the process $\{A_n(E_o), n \in Z\}$ is non-deterministic for some E_o, then inequality holds for Lebesgue a.e. value of E.

Our result here (theorem 1 below) attempts to subsume these results. It says that if equality $\gamma_+ = \gamma_-$ holds, then the measures on the projective space which are invariant under the action of the random sequence have to be "deterministic". Precise statement is in section I. At first sight, it does not seem very easy to use. We show how to deduce from it a further weakening of the independence condition in Furstenberg's criterion (corollary 1) and a direct proof of Kotani's result (stated as theorem 2, and proved in section IV).

The proof of theorem 1 relies upon an entropy estimate and considerations which are explained in section II. This entropy estimate now is a slight extension of a result in [L (section III.5)] and the proof is given in section III (see also [LY, sections 5 and 11)]).

Needles to say, we don't assert here that we generalize both Furstenberg's and Kotani's theories. We consider only a particular result of each theory, which in both cases is in fact a result in ergodic theory. The proof we give is based upon common ergodic ideas: entropy is smaller than exponents and entropy zero means deterministic. This scheme was suggested by a remark of Y. Derriennic, which initiated this paper.

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I - Notations and statement of results.

1.1 Main result.

In all the paper, $(\Omega, \mathcal{A}, m, \theta, A)$ is a model for a stationary biinfinite sequence of $d \times d$ real matrices, namely:

- $(\Omega, \mathcal{A}, m)$ is a Lebesgue space, i.e. a complete probability space, with the same Borel structure as the union of the unit interval and possibly a countable number of points (see [Ro])
- $\theta : \Omega \to \Omega$ is a one-to-one measure preserving measurable transformation,
- $A : \Omega \to GL(d,R)$ is a measurable random matrix such that

$$E \log(\max ||A||, ||A^{-1}||) < +\infty .$$

(We use the probabilistic notation Ef for the integral of a measurable real function f on Ω).

Set:

$$(1) \quad \begin{cases} A^{(n)}(\omega) = A(\theta^{n-1}\omega) \dots A(\omega) & \text{for } n > 0 \\ A^{(o)}(\omega) = Id & \\ A^{(n)}(\omega) = A^{-1}(\theta^n\omega) \dots A^{-1}(\theta^{-1}\omega) & \text{for } n < 0 \end{cases}$$

Then:

$$(2) \quad A^{(n+m)}(\omega) = A^{(m)}(\theta^n\omega)\, A^{(n)}(\omega) \quad \text{for all } n,m \in Z .$$

Set:

$$(3) \quad \begin{cases} \gamma_+ = \lim\limits_{n\to+\infty} \frac{1}{n} E \log ||A^{(n)}|| \\ \gamma_- = \lim\limits_{n\to+\infty} -\frac{1}{n} E \log ||(A^{(n)})^{-1}|| \end{cases}$$

where $||B||$ is the operator norm of a matrix B acting on the euclidean space R^d. The limits exist because the sequences are bounded and sub (or sur-)additive. Remark that, by stationarity, we have:

$$(4) \quad \begin{cases} \lim\limits_{n\to-\infty} \frac{1}{|n|} E \log ||A^{(n)}|| = -\gamma_- \\ \lim\limits_{n\to-\infty} -\frac{1}{|n|} E \log ||(A^{(n)})^{-1}|| = -\gamma_+ . \end{cases}$$

Remark also that, since $||B||\ ||B^{-1}|| \geqq 1$ for all $B \in GL(d,R)$, we have:

(5) $\gamma_+ \geqq \gamma_-$.

We want to study under which conditions equality $\gamma_+ = \gamma_-$ may hold.

Any two non-zero vectors of R^d are said to be equivalent if they are proportional. The space of equivalence classes is noted P^{d-1}, it is a compact connected manifold with dimension d-1. The action of a matrix A on R^d preserves the equivalence relation, we write again A for the quotient action on P^{d-1}. We shall consider only probability easure s on $\Omega \times P^{d-1}$, the marginal of which on Ω is m . Such a measure μ is identified with the essentially unique measurable function $\omega \to \mu_\omega$ from Ω into the set of probabilities on P^{d-1} such that

$$\mu(d\omega,ds) = \mu_\omega(ds)\cdot m(d\omega).$$

Define $\hat{\theta} : \Omega \times P^{d-1} \to \Omega \times P^{d-1}$ by:

$$\hat{\theta}(\omega,s) = (\theta\omega, A(\omega)\cdot s)$$

A measure μ on $\Omega \times P^{d-1}$ is invariant under $\hat{\theta}$ iff we have, for m-a.e. ω :

(6) $\mu_{\theta\omega} = A(\omega) \cdot \mu_\omega$

where we write $\nu \to B\cdot\nu$ for the action of a matrix B on the measures on P^{d-1} :

$$B\cdot\nu(f) = \int f(Bs)\, \nu(ds).$$

The set of invariant measures is a non-empty convex w*-compact set, where w* topology is defined by duality with the space $L^1(\Omega, C(P^{d-1}))$.

Let $\mathcal{B}$ and $\mathcal{C}$ be sub σ-algebras of A . We write $\mathcal{B} \subset \mathcal{C}$ if for every $B \in \mathcal{B}$ there exists a set $C \in \mathcal{C}$ with $m(B \Delta C) = 0$. We say $\mathcal{B}$ and $\mathcal{C}$ coincide if $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$. The σ-algebra $\mathcal{B}$ will be called decreasing if $\theta^{-1}\mathcal{B} \subset \mathcal{B}$. In this case write $\mathcal{B}_n$ for $\theta^{-n}\mathcal{B}$, $\mathcal{B}_\infty$ for $\bigcap_n \mathcal{B}_n$ and $\mathcal{B}_{-\infty}$ for the σ-algebra generated by all $\mathcal{B}_n$, $n \in Z$. Finally, if $\{X_i, i \in N\}$ are variables, write $\sigma(X_i, i \in N)$ for the σ-algebra generated by $X_i, i \in N$.

We can state now our main result.

Theorem 1: *Let* $(\Omega, A, m, \theta, A)$ *be a stationary sequence of matrices as above and* $\mathcal{B}$ *be a decreasing sub σ-algebra of* A *s.t.* $\sigma(A) \subset \mathcal{B}$. *Suppose*

a) $\gamma_+ = \gamma_-$

b) $\omega \to \mu_\omega$ *is a* $\hat{\theta}$*-invariant measure on* $\Omega \times P^{d-1}$ *s.t.* $\omega \to \mu_\omega$ is $\mathcal{B}_{-\infty}$ *- measurable,*

then, $\omega \to \mu_\omega$ *is* $\mathcal{B}$*-measurable.*

1.2 Generalization of Furstenberg's theorem.

When we apply theorem 1, we also consider the reversed stationary sequence $(\Omega, A, m, \theta^{-1}, A^{-1} \circ \theta^{-1})$. Invariant measures are the same ones (because the invariance relation (6) is the same). By (4), equality $\gamma_+ = \gamma_-$ also holds simultaneously. But sub σ-algebras which are decreasing under θ^{-1} are somehow orthogonal to the sub σ-algebras which are decreasing under θ. For instance, we can prove:

Corollary 1: *Let* $(\Omega, A, m, \theta, A)$ *be a stationary sequence of matrices such that*

i) there is no measure on P^{d-1} *invariant under m-a.e.* A *and*

ii) the σ*-algebra* $\sigma(A \circ \theta^n, n \geqq 0) \cap \sigma(A \circ \theta^n, n < 0)$ *coincides with the trivial* σ*-algebra* $\{\emptyset, \Omega\}$,

then $\gamma_+ > \gamma_-$.

Proof: Consider $\Omega \times P^{d-1}$ and $\omega \to \mu_\omega$ a $\hat{\theta}$-invariant measure. Replacing if necessary $\mu \to \mu_\omega$ by its conditional expectation

$$\omega \to E(\mu_. / \sigma(A \circ \theta^n, n \in Z)\ (\omega),$$

we may suppose that $\omega \to \mu_\omega$ is $(A \circ \theta^n, n \in Z)$ measurable.

If $\gamma_+ = \gamma_-$, by theorem 1 applied to $\mathcal{B} = \sigma(A \circ \theta^n, n \geqq 0)$, $\omega \to \mu_\omega$ is $\mathcal{B}$-measurable. By theorem 1 applied to the reversed sequence $(\Omega, A, m, \theta^{-1}, A^{-1} \circ O^{-1})$ and $\mathcal{B}' = \sigma(A \circ \theta^n, n < 0)$, $\omega \to \mu_\omega$ is $\mathcal{B}'$-measurable as well. By condition ii) $\omega \to \mu_\omega$ has to be an a.s. constant measure μ_o. Now, the invariance relation $\mu_o = A(\omega)\mu_o$ m-a.e. is a conflict with condition i).

□

If $A \circ \theta^n$ is a sequence of independent matrices, condition ii) is satisfied and corollary 1 reads as "i) ⇒ $\gamma_+ > \gamma$", a well known result of Fürstenberg [F].

In [LR] we showed that conditions i) and ii), together with some exponential mixing condition, imply $\gamma_+ > \gamma_-$. Here we remove the last assumption. We reformulate the result in the Markov case:

Corollary 2: Let (X,P) be a discrete Markov process with an invariant probability measure M, and let $A : X \to GL(d,R)$ satisfy

$$E \log \max(\|A\|, \|A^{-1}\|) < +\infty .$$

Let (Ω, A, m) be the space of trajectories of the Markov process with the canonical measure, θ be the shift transformation, A defined by $A(\{X_n, n \in Z\}) = A(X_o)$.

If $\gamma_+ = \gamma_-$, then there exists a measurable family $\{\pi(x), x \in X\}$ of probability measures on P^{d-1} such that for M-a.e. $x \in X$:

$$\pi(y) = A(x)\cdot\pi(x) \quad \text{for } P(x,\cdot)\text{-a.e. } y .$$

This result is essentially due to Virtser [V], Guivarc'h [G] and Royer [R]. Our proof is the same as the proof of corollary 1, but here, by the Markov property, we have

$$\sigma(X_n, n < 0) \cap \sigma(X_n, n \geqq 0) \subset \sigma(X_o) \quad \text{instead of ii).}$$

1.3 A result in Kotani's theory.

Consider $(\Omega, A, m, \theta, A_E)$ as above, where $A_E = \begin{pmatrix} -V(\omega)+E & -1 \\ 1 & 0 \end{pmatrix}$, $V(\omega)$ is a measurable real function on Ω, satisfying $E \log(\max(|V(\omega)|, 1)) < +\infty$, and E is a real parameter.

Fix E. Since $\det A_E = 1$, the exponents γ_+ and γ_- are opposite. Write $\gamma(E)$ for their common absolute value. By (5), $\gamma(E) \geqq 0$ and the question is whether $\gamma(E)$ is positive.

Theorem 2 ([S] theorem 3): Suppose the σ-algebras $\sigma(V \circ \theta^n, n \in Z)$ and $\bigcap_m \sigma(V \circ \theta^n, n \geqq m)$ do not coincide. Then, $\gamma(E) > 0$ for Lebesgue a.e. E.

We prove theorem 2 in section IV. The proof uses theorem 1 and elementary properties of the associated difference operator.

II - Non-invertible case.

2.1 Notations revisited. Entropy.

Let $(X, \mathcal{E}, P)$ be a Lebesgue space, $T : X \to X$ a measurable measure-preserving transformation. We do not insist now on T being invertible. In general, there exists a probability transition kernel $P(x,\cdot)$ defined on $X \times \mathcal{E}$ such that for any f,g positive measurable functions

$$E(Pf \cdot g) = E(f \cdot g \circ T) \tag{7}$$

where $Pf(x) = \int f(y) P(x,dy)$.

With probability 1, the measure $P(x,\cdot)$ is carried by the set of preimages of x. Let $A : X \to Gl(d,R)$ be a measurable random matrix, satisfying

$$E \log \max(\|A\|, \|A^{-1}\|) < \infty .$$

Define $\{A^{(n)}, n \geqq 0\}$ by (1), γ_+ and γ_- by (3).

Consider the space P^{d-1} and the transformation $\hat{T}$:

$$\hat{T}(x,s) = (Tx, A(x,s)).$$

By a measure on $X \times P^{d-1}$ we again mean a probability measure, the marginal of which on X is P. The set of $\hat{T}$-invariant measures on $X \times P^{d-1}$ is a non-empty convex w* compact set. Invariance formula now reads as follows

Proposition 1: Let $x \to \nu_x$ be a $\hat{T}$-invariant measure on $X \times P^{d-1}$. Then, for P-a.e. x

$$\nu_x = \int A(y)\cdot\nu_y \; P(x,dy). \tag{8}$$

Proof: Consider a positive function $F(x,s) = g(x)h(s)$ and a $\hat{T}$-invariant measure ν on $X \times P^{d-1}$. We have:

$$\int F \circ \hat{T}\, d\nu = E(g \circ T \cdot H) = E(g\ PH) \quad \text{by (7)}$$

where $H(x) = \int h(A(x)s)\ \nu_x(ds)$.

Relation (8) thus follows from $\int F \circ \hat{T}\, d\nu = \int F\, d\nu$ for every F.

□

We define now the entropy α_ν of a $\hat{T}$-invariant measure ν by:

$$(9) \qquad \alpha_\nu(\hat{T}) = -E\left(\int \log \frac{dA^{-1}(x)\nu_{Tx}}{d\nu_x}(s)\ \ \nu_x(ds)\right)$$

where $\frac{d\mu}{d\nu}(s)$ denotes the (essentially) unique function f such that

$$\mu = f\nu + \mu', \text{ with } \mu' \text{ and } \nu \text{ mutually singular.}$$

The number α_ν is an average Küllback information (see [K]). From Jensen's inequality follows:

Proposition 2: We have $\alpha_\nu \geqq 0$. *Equality* $\alpha_\nu = 0$ *holds iff for P-a.e.* x

$$A(x) \cdot \nu_x = \nu_{Tx},$$

2.2 A relation between entropy, exponents and dimension.

In this subsection, we state our key result. It relates the exponents and the entropy.

Theorem 3: Let $(X,\mathcal{E},P,T,A)$ *be as in section 2.1,* ν *a* $\hat{T}$*-invariant measure on* $X \times P^{d-1}$, *then:*

$$\alpha_\nu \leqq (d-1)\ (\gamma_+ - \gamma_-)\ .$$

Theorem 3 is a simple consequence of

Proposition 3: Let $(X,\mathcal{E},P,T,A,\nu)$ *be as above, then*

$$\alpha_\nu \leqq (d-1)\ E(\log||A|| + \log||A^{-1}||)\ .$$

In fact, applying proposition 3 to $(X,\mathcal{E},P,T^n,A^{(n)},\nu)$ yields for each $n > 0$:

$$\alpha_\nu(\hat{T}^n) \leqq (d-1)\ E(\log||A^{(n)}|| + \log||(A^{(n)})^{-1}||)\ .$$

Theorem 3 follows, using that for $n > 0$

$$\alpha_\nu(\hat{T}^n) = n\ \alpha_\nu(\hat{T})$$

and the definition (3) of γ_+ and γ_-.

Consider $x \to \nu_x$ a measurable family of probabilities on P^{d-1}. We shall define globally a dimension of such an object, which is in fact smaller than some dimension of almost every metric probability space (P^{d-1},ν_x). For $\varepsilon > 0$, define $p_\varepsilon(x,s) : X \times P^{d-1} \to R_+$ by $p_\varepsilon(x,s) = \dfrac{\log \nu_x B(s,\varepsilon)}{\log \varepsilon}$ where $B(s,\varepsilon)$ is the open ball of radius ε for the natural angular metric on P^{d-1}. For $\chi > 0$, set:

$$(10) \quad \left\{ \begin{aligned} &\beta(\chi) = \sup\{t : t \in R,\ \exists \varepsilon(t) \text{ s.t. for all } 0 < \varepsilon < \varepsilon(t)\ ,\\ &\qquad\qquad \nu\{(x,s) : p_\varepsilon(x,s) > t\} \geqq 1-\chi\} \\ &\text{and} \\ &\qquad \dim \nu = \lim_{\chi \to 0} \beta(\chi) \end{aligned} \right.$$

We prove in section III the following estimates:

Proposition 4: Let $(X,\mathcal{E},P,T,A,\nu)$ *be as above and suppose* ν *is* $\hat{T}$*-ergodic. Then*

$$(11) \qquad \dim \nu \leqq d-1$$

$$(12) \qquad \alpha_\nu \leqq \dim \nu \cdot E(\log||A|| \cdot ||A^{-1}||);$$

Proposition 3 follows clearly from propostion 4 if the invariant measure ν is $\hat{T}$-ergodic. In general write $\nu = \int_\xi \nu\, d\xi$ for the ergodic decomposition of ν. Then, at P-a.e. x, $\nu_x = \int_\xi \nu_x\, d\xi$. Since at ${}_\xi\nu$-a.e. (x,s) :

$$\frac{dA^{-1}(x)\quad {}_{Tx}}{d\nu_x}(s) = \frac{dA^{-1}(x)\,{}_{\xi}\nu_{Tx}}{d_{\xi}\nu_x}(s),$$

we have:

$$\int \log \frac{dA^{-1}(x)\ \nu_{Tx}}{d\nu_x}(s)\ {}_{\xi}\nu_x(ds) = \int \log \frac{dA^{-1}(x)\,{}_{\xi}\nu_{Tx}}{d_{\xi}\nu_x}(s)\ {}_{\xi}\nu_x(ds)$$

and proposition 3 follows in full generality.

2.3 Proof of theorem 1.

We now prove that theorem 3 implies theorem 1. Consider $(\Omega, A, m, \Theta, A)$ as in section I, and B a decreasing sub σ-algebra s.t. A is B-measurable. Since (Ω, A, m) is a Lebesgue space, there exists a projection $\Pi : \Omega \to X$, where (X, E, P) is a Lebesgue space, such that the σ-algebras $\Pi^{-1}E$ and B coincide (see e.g. [Ro] for the construction of the quotient space (X, E, P)). Since B is decreasing, there exists a measurable map $T : X \to X$ s.t. $\Pi\Theta = T\Pi$ m-a.s. Clearly, T preserves the measure $P = m \circ \Pi^{-1}$. The map $A : \Omega \to Gl(d,R)$ is B-measurable and thus factorizes through Π. We write again A for the quotient map, $A : X \to GL(d,R)$. Exponents γ_+ and γ_- are the same for both systems $(\Omega, A, m, \Theta, A)$ and (X, E, P, T, A). Define:

$$\hat{\Pi} : \Omega \times P^{d-1} \to X \times P^{d-1} \quad \text{by} : \quad \hat{\Pi}(\omega, s) = (\Pi(\omega), s).$$

Then, for m-a.e. ω, $\hat{\Pi} \cdot \hat{\Theta}(\omega, s) = \hat{T}\,\hat{\Pi}(\omega, s)$ for all s.

If μ is a $\hat{\Theta}$-invariant measure, then $\nu = \mu \circ \Pi^{-1}$ is a $\hat{T}$-invariant measure. Let C be a sub σ-algebra of A, μ a measure on $\Omega \times P^{d-1}$. We write $\omega \to E(\mu_{\bullet}/C)(\omega)$ for the unique measure on $\Omega \times P^{d-1}$ which is C-measurable and yields the same integrals as μ.

If the measure μ is $\hat{\Theta}$-invariant and the σ-algebra C is Θ-invariant (i.e. $\Theta^{-1}C$ and C coincide) then the measure $E(\mu/C)$ is $\hat{\Theta}$-invariant.

Lemma 1: *Let* $\omega \to \mu_\omega$ *be a* $\hat{\Theta}$-*invariant measure and write* $x \to \nu_x$ *for the* $\hat{T}$-*invariant measure* $\nu = \mu \cdot \hat{\Pi}^{-1}$. *Then, for m-a.e.* ω :

$$(13) \qquad E(\mu_{\bullet}/B_{-\infty})(\omega) = \lim_{n\to+\infty} A(\Theta^{-1}\omega) \ldots A(\Theta^{-n}\omega)\ \nu_{\Pi(\Theta^{-n}\omega)} .$$

Proof of lemma 1: Since $x \to \nu_x$ is invariant, by (8) the sequence $A(\Theta^{-1}\ ) \ldots A(O^{-n}\omega)\ \nu_{\Pi(\Theta^{-n}\omega)}$ is a B_{-n}-martingale of probability measures. Write μ''_ω for the limit a.e. Then, $\omega \to \mu''_\omega$ is $\hat{\Theta}$-invariant ((6) is satisfied), $\omega \to \mu''_\omega$ is $B_{-\infty}$-measurable, and $\mu'' \circ \hat{\Pi}^{-1} = \nu$. These properties characterize $E(\mu_{\bullet}/B_{-\infty})$.

□

Now suppose $\gamma_+ = \gamma_-$ and μ is a $\hat{\Theta}$-invariant measure. Let $\nu = \mu \circ \hat{\Pi}^{-1}$. By theorem 3, $\alpha_\nu = 0$ and proposition 2 yields $A(x)\ \nu_x = \nu_{Tx}$ at P-a.e. x. Therefore we have at m-a.e. ω, for all $k > 0$,

$$\begin{aligned} A(\Theta^{-k}\omega)\,\nu_{\Pi(\Theta^{-k}\omega)} &= \nu_{T\Pi\Theta^{-k}\omega} \\ &= \nu_{\Pi\Theta^{-k+1}\omega} . \end{aligned}$$

Transfering to formula (13), we get at m-a.e. ω

$$E(\mu_{\bullet}/B_{-\infty})(\omega) = \lim_{n\to+\infty} \nu_{\Pi(\omega)} = \nu_{\Pi(\omega)} .$$

If furthermore $\omega \to \mu_\omega$ is $B_{-\infty}$-measurable, then $E(\mu_{\bullet}/B_{-\infty})(\omega) = \mu_\omega$ m-a.e. and theorem 1 follows.

Summarizing sections 2.2 and 2.3, theorem 1 is a consequence of proposition 4, which we now proceed to prove.

III - Proof of proposition 4.

3.1 Geometrical properties.

Consider a metric space S, locally Lipeomorphic to the open unit ball in R^k. For $s \in S$, let $B(s,r)$ denote the ball of radius r centered at s. Balls in S have the same topological properties as the subsets of R^k for which the ratio of the outer diameter and the inner diameter is bounded. In particular.

Besicovitch Covering Lemma (see e.g. [Gu] page 2).

Let S be a compact metric space, locally Lipeomorphic to the open unit ball in R^k. There exists a number $C(S)$ such that if E is a subset of S and $r : E \to (0,\infty)$, there exists a subcover A' of the cover A

$$A = \{B(s,r(s)), s \in E\} \quad \text{of} \quad E$$

such that no element s in S lies in more than $C(S)$ elements of A'.

Proposition 5: *Let S be a compact metric space locally Lipeomorphic to the open unit ball in R^k. Let ν be a probability measure on S. Then:*

$$\limsup_{\delta \to 0} \frac{\log \nu B(s,\delta)}{\log \delta} \leqq k \qquad \nu\text{-a.e.}$$

If μ is another probability measure on S,

$$\lim_{\delta\to 0} \frac{\mu B(s,\delta)}{\nu B(s,\delta)} = \frac{d\mu}{d\nu}(s) \qquad \nu\text{-a.e.}$$

and

$$\int \log f^* \, d\nu \leqq C(S) ,$$

where $f^*(s) = \max(\frac{\mu B(s,\delta)}{\nu B(s,\delta)}, \delta > 0)$ *and $C(S)$ is the constant in BCL.*

The proof of proposition 5 is the same as the classical proofs of the corresponding statements pertaining to the Lebesgue measure on R^k. One uses BCL instead of Vitali's lemma.

We now consider (X,E,P,T,A) as in section II and $x \to \nu_x$ a $\hat{T}$-invariant measure on $X \times P^{d-1}$. For P-a.e. x we apply proposition 5 to the probability measures ν_x and $A^{-1}(x)\nu_{Tx}$ on the compact manifold P^{d-1}. Integrating with respect to P, we obtain:

$$(17) \qquad \limsup_{\delta \to 0} \frac{\log \nu_x B(s,\delta)}{\log \delta} \leqq d-1 \quad \text{for} \quad \nu\text{-a.e.} \quad (x,s)$$

$$(18) \qquad \lim_{\delta\to 0} \frac{\nu_{Tx}(A(x)B(s,\delta))}{\nu_x B(s,\delta)} = \frac{dA^{-1}(x)\nu_{Tx}}{d\nu_x}(s) \qquad \nu\text{-a.e.}$$

and

$$(19) \qquad \int \log f^* \, d\nu \leqq C_d , \quad \text{where}$$

$$f^*(x,s) = \max(\frac{\nu_{Tx}(A(x)\ B(s,\delta))}{\nu_x\ B(s,\delta)} , \delta > 0)$$

and C_d is a constant depending only on d.

The definition (10) of $\dim_\nu$ and (17) imply (11). Before using (18) and (19) to prove (12) we have another geometrical lemma. For a matrix $A \in Gl(d,R)$, we write

$$\phi(A) = \frac{1}{||A||\ ||A^{-1}||} .$$

By definition $0 < \phi \leqq 1$.

Lemma 2: *Fix $\varepsilon > 0$. There exists $\delta_o > 0$ such that if $\delta \leqq \delta_o$, then for all matrices $A \in GL(d,R)$, all $s \in P^{d-1}$,*

$$A\ B(s,\delta) \supset B(As, \delta \cdot \phi(A) e^{-\varepsilon}) .$$

Proof of lemma 2: The property is easy to prove for diagonal matrices with positive entries. Write any matrix A as $A = K_1 \Delta K_2$ where K_1 and K_2 are orthogonal and Δ is diagonal with positive entries. Since K_1 and K_2 act as isometries on P^{d-1}, the lemma follows.

□

We summarize in the following proposition the consequences of the results of this section.

Proposition 6: *Fix $\varepsilon > 0$ and consider $x \to \nu_x$ a $\hat{T}$-invariant measure on $X \times P^{d-1}$.*

There exist a number $\delta_o(\varepsilon)$ and a measurable real function $h_\varepsilon(x,s)$, $h_\varepsilon > 0$ ν-a.e., such that: If $0 < \delta \leqq h_\varepsilon(x,s)$

$$\frac{\nu_{Tx} B(A(x)s, \delta\phi(A(x))e^{-\varepsilon})}{\nu_x B(s,\delta)} \leqq e^{\varepsilon} \quad (x,s) \quad \chi_{\{f>0\}} + e^{-\frac{1}{\varepsilon\nu(f=0)}} \chi_{\{f=0\}}$$

and if $0 < \delta \leqq \delta_0(\varepsilon)$,

$$\frac{\nu_{Tx} \; B(A(x)s, \delta\phi(A(x)e^{-\varepsilon})}{\nu_x \; B(s,\delta)} \leqq f^*(x,s)$$

where $\quad f(x,s) = \dfrac{dA^{-1}(x) \; \nu_{Tx}}{d\nu_x}(s)$ *and* $f^* \geqq 1$, $\int \log f^* \, d\nu < +\infty$.

3.2 End of the proof.

Proposition 7: Fix $\varepsilon > 0$ *and consider* $x \to \nu_x$ *a* $\hat{T}$*-invariant ergodic measure on* $X \times P^{d-1}$. *Then, at* ν*-a.e.* (x,s)

$$\liminf_n -\frac{1}{n} \log \nu_{T^n x} B(A^{(n)}(x)s, \; \delta_0(\varepsilon)e^{-n\varepsilon} \prod_{i=0}^{n-1} \phi(A(T^i x)))$$

$$\geqq \alpha_\nu - 3\varepsilon \qquad \textit{if} \quad \alpha_\nu < \infty \,,$$

$$\geqq \frac{1}{\varepsilon} \qquad \textit{if} \quad \alpha_\nu = \infty \,,$$

where $\delta_0(\varepsilon)$ *is chosen in proposition 6* .

<u>Proof:</u> We write $\nu_{T^n x} B(A^{(n)}(x)s, \; \delta_0(\varepsilon)e^{-n\varepsilon} \prod_{i=0}^{n-1} \phi(A(T^i x)))$ as

$$\nu_x \; B(s, \delta_0(\varepsilon)) \prod_{j=0}^{n-1} q_j(x,s) \;,$$

where

$$q_j(x,s) = \frac{\nu_{T^{j+1}x} \; B(A^{(j+1)}(x)s, \; \delta_0(\varepsilon)e^{-(j+1)\varepsilon} \prod_{i=0}^{j} \phi(A(T^i x)))}{\nu_{T^j x} \; B(A^{(j)}(x)s, \; \delta_0(\varepsilon) \; e^{-j\varepsilon} \prod_{i=0}^{j-1} \phi(A(T^i x)))}$$

and we shall estimate $q_j(x,s)$ by proposition 6 .

Choose δ small enough such that

$$\delta \leqq \delta_0(\varepsilon) \;, \quad \int_{\{h_\varepsilon < \delta\}} \log f^* \, d\nu \leqq \varepsilon \quad \text{and if } f \text{ positive a.e.,}$$

$$-\int_{\{h_\varepsilon \geqq \delta\}} \log f \, d\nu \geqq \alpha_\nu - \varepsilon \qquad \text{if} \quad \alpha_\nu < \infty$$

$$-\int_{\{h_\varepsilon \geqq \delta\}} \log f \, d\nu \geqq \frac{1}{\varepsilon} + 2\varepsilon \qquad \text{if} \quad \alpha_\nu = \infty \,.$$

For $j > -\dfrac{\log \delta/\delta_0(\varepsilon)}{\varepsilon} = N$, we have $\delta_0(\varepsilon)e^{-j\varepsilon} \prod_{i=0}^{j-1} \phi(A(T^i x)) < \delta \leqq \delta_0(\varepsilon)$ and we can apply proposition 6 at the point $(T^j x, A^{(j)}(x)s)$ - which is $\hat{T}^j(x,s)$ - and thus estimate $q_j(x,s)$ by either $q_j(x,s) \leqq e^{\varepsilon} f(\hat{T}^j(x,s))$ if $\delta \leqq h_\varepsilon(\hat{T}^j(x,s))$ or $q_j(x,s) \leqq f^*(\hat{T}^j(x,s))$ if $\delta > h_\varepsilon(\hat{T}^j(x,s))$.
Therefore:

$$\log \prod_{j=0}^{n-1} q_j \leqq \prod_{j=0}^{N} \log q_j + n\varepsilon + \sum_{j=N+1}^{n-1} g \circ \hat{T}^j \;,$$

where g is given by:

$$\begin{aligned} g(x,s) &= \log f(x,s) && \text{if} \quad h_\varepsilon(x,s) \geqq \delta \,, \quad f(x,s) > 0 \\ &= -\frac{1}{\varepsilon\nu(f=0)} && \text{if} \quad h_\varepsilon(x,s) \geqq \delta \,, \quad f(x,s) = 0 \,, \\ &= \log f^*(x,s) && \text{otherwise.} \end{aligned}$$

Proposition 7 follows by the Birkhoff ergodic theorem and our choice of δ.

□

We now prove proposition 4. Relation (11) follows from (17). Besides, fix $\varepsilon, \chi > 0$. If n is large enough, we have $\nu(D_1) \geqq 1 - \chi/_2$, where

$$D_1 = \{(x,s) \; : \; \prod_{i=0}^{n-1} \phi(A(T^i x)) \geqq e^{-n \; E \; \log(||A|| \; ||A^{-1}||)} e^{-n\varepsilon}\} \;.$$

By proposition 7, if n is large enough, we thus have $\nu(D_2) \geqq 1 - \chi$, where

$$D_2 = \{(x,s) \; : -\frac{1}{n}\log \nu_{T^n x} B(A^{(n)}(x)s, \delta_0(\varepsilon)e^{-2n\varepsilon} e^{-nE \; \log(||A|| \; ||A^{-1}||)})$$

$$\geqq \alpha_\nu - 3\varepsilon \qquad \text{if} \quad \alpha_\nu < \infty$$

$$\geqq \frac{1}{\varepsilon} \qquad \text{if} \quad \alpha_\nu = \infty \} \,.$$

By the $\hat{T}$-invariance of ν and the definition (10), we have:

$$\begin{aligned}(E \log(||A||\ ||A^{-1}||) + 2\varepsilon)\beta(\chi) &\geqq \alpha_\nu - 3\varepsilon && \text{if } \alpha_\nu < \infty \\ &\geqq \frac{1}{\varepsilon} && \text{if } \alpha_\nu = \infty .\end{aligned}$$

Letting χ be arbitrarily small, we have the same relation with $\dim \nu$. Since $\dim \nu \leqq d-1 < +\infty$, and ε is arbitrary, we cannot have $\alpha_\nu = \infty$ and relation (12) follows.

IV - Application to Jacobi matrices.

4.1 The difference operator.

In this subsection, we recall several simple facts related to the difference operator associated with a product of Jacobi matrices (see [S]). Let (Ω, A, m, θ) be a dynamical system, and $V : \Omega \to R$ a measurable function. For each $\omega \in \Omega$ define the operator H_ω^- on $\ell^2(Z \setminus N)$ by:

$$\begin{cases} H_\omega^- u(n) = u(n+1) + u(n-1) + V(\theta^n\omega)u(n) & \text{for all } n < 0 \\ \quad \text{where } u(0) = 0 . \end{cases}$$

Then, H_ω^- is a self-adjoint operator on $\ell^2(Z \setminus N)$ with domain

$$\mathcal{D} = \{u(n), n < 0 : \sum_{n<0} |u(n)|^2 < +\infty ,\ \sum_{n<0} |H_\omega^- u(n)|^2 < +\infty\} .$$

Write ρ_ω for the spectral measure of the function δ_{-1}. By definition, ρ_ω is a probability measure on R such that for any Borel complex function ϕ on R, we have:

$$\langle\phi(H_\omega^-)\delta_{-1}, \delta_{-1}\rangle = \int\phi(\lambda)\rho_\omega(d\lambda) . \tag{20}$$

We shall use (20) to prove:

Proposition 8: Let (Ω, A, m, θ) be a dynamical system, $V : \Omega \to R$ a measurable function. Then, there exists at each ω a class of complex functions q on R such that $\mathrm{Im}\, q \geqq 0$ and:

a) $(\omega, E) \to q(\omega, E)$ is $\sigma(E, V \circ \theta^n, n < 0)$-measurable,

b) if $q(\omega,E)$ $q(\omega',E)$ for a set of E of positive Lebesgue measure, then $V(\theta^{-1}\omega) = V(\theta^{-1}\omega')$, and

c) at each ω, for Lebesgue a.e. E, we have:

$$q(\theta\omega, E) = -V(\omega) + E - \frac{1}{q(\omega,E)} .$$

<u>Proof:</u> Fix $\omega \in \Omega$ and define for $\mathrm{Im}\, z > 0$

$$m_-(\omega,z) = \langle(H_\omega^- - z)^{-1}\delta_{-1}, \delta_{-1}\rangle \quad \text{and} \quad q(\omega,E) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{-1}{m_-(\omega, E+i\varepsilon)}$$

when the limit exists.

By (20) we have $m_-(\omega,z) = \int\frac{\rho_\omega(d\lambda)}{\lambda - z}$ and therefore $z \to m_-(\omega,z)$ is a Herglotz function. By Fatou's theorem for Herglotz functions (see in our case [Ko], [S]), $q(\omega,E)$ is defined for Lebesgue a.e. E, $\mathrm{Im}\, q(\omega,E) \geqq 0$, and if $q(\omega,E) = q(\omega',E)$ for a set of E of positive Lebesgue measure, then $\rho_\omega = \rho_{\omega'}$.

By (20), it is easy to see that $\omega \to \rho_\omega$ is $\sigma(V \circ \theta^n, n < 0)$-measurable. Then property a) follows from the definition of q. Also by (20) we have

$$\int\lambda\rho_\omega(d\lambda) = \langle H_\omega^- \delta_{-1}, \delta_{-1}\rangle = V(\theta^{-1}\omega)$$

and this yields property b).

Write $\{u(n), n < 0\}$ for $(H_\omega^- - z)^{-1}\delta_{-1}$, we have : $u(-1) = m_-(\omega,z)$

$$\begin{aligned} &u(-2) + (V(\theta^{-1}\omega) - z)\, u(-1) = 1 \\ &u(n-1) + u(n+1) + (V(\theta^n\omega) - z)\, u(n) = 0 \quad \text{for all } n < -1 . \end{aligned} \tag{21}$$

Furthermore $m_-(\omega,z)$ is by definition the only complex value of $u(-1)$ such that (21) yields a sequence $\{u(n), n < 0\}$ in ℓ^2.

Write $\{w(n), n < 0\}$ for $(H_{\theta\omega}^- - z)^{-1}\delta_{-1}$, and compare with (21).

We get $-\frac{w(-2)}{w(-1)} = m_-(\omega,z)$, i.e.

$$-\frac{1}{m_-(\theta\omega,z)} + V(\omega) - z = m_-(\omega,z) .$$

Property c) follows for all E such that $q(\omega,E)$ and $q(\theta\omega,E)$ exist.

4.2 Proof of theorem 2.

Let $(\Omega,\mathcal{A},m,\theta)$ be a dynamical system and V a real measurable function such that

$$E \log \max(|V|,1) < +\infty .$$

Define

$$A_E(\omega) = \begin{pmatrix} -V(\omega) + E & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma(E) = \lim_{n\to\infty} \frac{1}{n} E(\log\|A_E^{(n)}\|) .$$

Write $\mathcal{B}_n$ for $\sigma(V \circ \theta^m, m \geqq n)$, $\mathcal{B}_\infty$ for $\bigcap\limits_n \mathcal{B}_n$, $\mathcal{B}_{-\infty}$ for $\sigma(V \circ \theta^n, m \in Z)$. Suppose $\gamma(E) = 0$ for E in a set Λ of positive Lebesgue measure, we want to show that the σ-algebras $\mathcal{B}_\infty$ and $\mathcal{B}_{-\infty}$ coincide. By stationarity, we only have to show that $\sigma(V \circ \theta^{-1}) \subset \mathcal{B}_0$. We may suppose Λ bounded.

We apply theorem 1 to the following sequence of matrices

$$(\Omega \times \Lambda,\ \mathcal{A} \times \mathcal{R},\ m \otimes \frac{dE}{|\Lambda|},\ \tilde{\theta}, A)$$

where $\mathcal{R}$ is the σ-algebra of Borel subsets of Λ ,

$$\tilde{\theta}(\omega,E) = (\theta\omega,E) \ , \quad A(\omega,E) = A_E(\omega) \ .$$

Clearly $\gamma_+ = \gamma_- = 0$. Let $\tilde{\mathcal{B}} = \mathcal{B}_0 \otimes \mathcal{R}$.

The σ-algebra $\tilde{\mathcal{B}}$ is decreasing and $\sigma(A) \subset \tilde{\mathcal{B}}$. The conclusion of theorem 1 is that any invariant measure on $\Omega \times \Lambda \times P^1$ which is $\tilde{\mathcal{B}}_{-\infty}(=\mathcal{B}_{-\infty} \otimes \mathcal{R})$ measurable is $\tilde{\mathcal{B}}_0$-measurable. If the projective space P^1 is identified with $R \cup \{\infty\}$ by $\binom{X}{Y} \leftrightarrow X/_Y$, the invariance relation (6) is satisfied if for a.e. (ω,E) the measure $\mu_{(\theta\omega,E)}$ is obtained from $\mu_{(\omega,E)}$ by applying the transformation

$$s \to -V(\omega) + E - \frac{1}{s} .$$

The central observation now is that proposition 8 (a) and c)) says that the Poisson measure $\mu_{(\omega,E)}$ of the point $q(\omega,E)$ defines a $\tilde{\mathcal{B}}_{-\infty}$ measurable invariant measure. By theorem 1, $(\omega,E) \to q(\omega,E)$ is $\mathcal{B}_0 \otimes \mathcal{R}$ measurable. By property b) in proposition 8, and the independence of $\mathcal{A}$ and $\mathcal{R}$, it follows that

$$\sigma(V \circ \theta^{-1}) \subset \mathcal{B}_0 \ , \quad \text{q.e.d.}$$

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