Jiagang YANG — Continuation of entropy

GdT "Dyn Cocycles"

Pesin theory in 70s

Paper by Newhouse in 80s

<u>Rem</u> h as a function of map, measure, both.

USC of entropy from smoothness

upper semi continuity ok for C^∞ (Yomdin)

ok for uniform hyperbolicity

fail for C^r, $r<\infty$ (non hyperbolic)

- C^r perturbation of homoclinic intervals

$\rightarrow h_{top}$ (small horseshoe) can approach $\frac{\text{min exponent}}{\text{differentiability}}$

<u>Conclusion</u> failure of USC of h_{top} in C^r ($r<\infty$).

- Similar for h_{top}: not USC in C^r

(for instance, Young: h_{top} (2-d flow) $=0$)

USC of entropy from expansiveness

Bowen's (asymptotic) h-expansiveness.

$\hookrightarrow \exists\, \varepsilon_0>0$ $h_{top}(B_\infty(x,\varepsilon_0))=0$

$\hookrightarrow \lim_{\varepsilon\to 0}\sup_x h_{top}(B_\infty(x,\varepsilon))=0$.

<u>Rem</u> implies USC of metric entropy

for top entropy, need local uniformity.

<u>Rem</u> pw affine surface diffeo (recent (3 years ago))

counter example by student of Newhouse

Def

f is ε-entropy expansive if $h^*_\varepsilon(f) = \sup_x h_{top}(f, B_\infty(x,\varepsilon))$
$= 0$

f is ε-C^r robustly entropy expansive if $\exists$ a neighborhood of f in C^r topology where $h^*_\varepsilon(\cdot) = 0$

f is asymptotically entropy expansive if ...

f is robustly ... if the previous convergence holds uniformly on a neighborhood of f.

Bowen: $h_\mu(f, \mathcal{A}) \le h_\mu(f) \le h_\mu(f, \mathcal{A}) + h^*_\varepsilon(f)$
if diam $\mathcal{A} \le \varepsilon$.

(this shows usc under expansivity properties).

Thm (Liao, Viana, J. Yang)
in $\mathrm{Diff}^1(M)$: away from tangencies, robust entropy expansiveness holds.

Corollaries
1) h_{top} is usc on $\mathrm{Diff}^1(M) \setminus \overline{HT^1(M)}$

2) Shub's conjecture away from tangencies
$$h_{top}(f) \ge \log \rho(f_*)$$
← total action on homology

asked for Lipschitz but false

$(\theta \mapsto 2\theta)$
φ goes to 0

Yomdin C^∞ ok
fails for C^1

Xia, Saghin for PH $d_c = 1$

proof: $\rho(f_*)$ is locally constant.

f_n C^∞ $\xrightarrow{C^1}$ f

$h_{top}(f_n) \geq \rho(f_{n*}) = \rho(f_*)$

usc yields the conclusion.

papers by Newhouse-Young, Yang & Crovisier, Pacifico-Díaz-Fisher-Vieitez

Proof of the theorem

① Lemma

f is entropy-expansive iff $\forall \mu \in \mathcal{M}_{inv}(f)$ for μ-a.e x $h^*_\varepsilon(f, B_\infty(x,\varepsilon)) = 0$

Rem a more general statement is true (replace 0 by any number).

② (Crovisier, Y) on the support of any ergodic inv proba, there is a PH splitting (with $d_c \leq 1$). [away from tangencies]

based on Mañé's ergodic lemma & Wen: far from tangencies, domination on periodic points.

for center-bundle splitting into 1-d (with domination)

This holds for open & dense set away from tangencies (Crovisier, Sambarino, Dawei Yang)

Metric entropy of volume preserving diffeos

Thm h_{vol} is continuous on a generic subset of $\mathrm{Diff}^1(M)$

Cor Pesin formula for C^1 diffeo.

Cor $\lambda_i(f)$ is generically continuous in $L^1(vol)$

Bochi Viana: For generic f,
on a.e. x there is a dominated splitting corresponding to the exponents.
$\to$ $B_\infty(x, \varepsilon_x)$ is contained in a mfd tangent to E^c for some $\varepsilon_x > 0$ depending on x

$\to$ $h_{top}(B_\infty(x, \varepsilon_x))$ is small (ε_x reduced if necessary)

Gives Λ_f with $Leb(\Lambda_f) \simeq 1$ where the above holds with some uniformity.

Avila-Bochi: (observation)
$C > 0$ $\lambda < 1$
$\forall f \in Diff^1_{vol}(M)$ $\Lambda_f := \{x : \exists$ d.s. on orb(x) where $\prod \|Df|_{E^{cs}(f^k_x)}\| \le C\lambda^n$

$\prod \|Df^{-1}|_{E^{cu}(f^{-k}_x)}\| \le C\lambda^n\}$

$\forall n \ge 0$

$f_n \xrightarrow{C^1} f$

by uniformity $\overline{\lim}\, \Lambda_{f_n} \subset \Lambda_f$

$\Rightarrow \overline{\lim}\, Leb(\Lambda_{f_n}) \le Leb(\Lambda_f)$

semicontinuity $\Rightarrow$ continuous on generic subset.

Applications

1) C^2 transitive diffeos with unique SRB μ_f.
we have stochastic stability (weak star) in C^2 topology.
This extends to C^2 $f_n \xrightarrow{C^1} f$ C^2
uses thermodynamical formalism & uniqueness

Proof

$P(\mu_n) = 0$

Consider an accumulation point μ_0 in weak *

USC $\Rightarrow P(\mu_0) \geq 0 \Rightarrow P(\mu_0) = 0$

$\Rightarrow \mu_0$ SRB.

2) C^1 generically transitive Anosov
(see also Hao Qiu) Quas on S^1.

Keller: $\underbrace{\#\{\mu : P_f(\mu) \geq 0\} = 1}_{(U)}$ then this is ~~SRB~~ physical (& unique)

For f C^2, (U) is known.

For f C^1, $P(f) \geq 0$

Consider $\mathcal{E}: f \mapsto E_f := \{\mu : P(\mu) \geq 0\}$

$\mathcal{E}$ is upper semi continuous, therefore continuous in the Hausdorff topology.

$\# E_f = 1$ if C^2 (so dense)

Therefore $\# E_f = 1$, i.e (U), holds on a generic subset.

Mostly contracting diffeos C^2

$E^{cs} \oplus E^u$ (def of Bonatti-Viana)

~~Any D_0 with positive volume~~
any disk in F^u contains a set of positive volume

$\Longleftrightarrow$ For any u-Gibbs μ $\lambda^{cs}(\mu) < 0$ | $\lim \frac{1}{n} \log \|Df^n|_{cs}\| < 0$

BV $\Rightarrow$ finitely many physical measures (*)
st $\bigcup$ basins $=$ M mod 0.

- the condition "mostly contracting" is C^2 open
  (M. Andersson)

- It is actually C^1 open (among C^2)

  ↳ PH, $d_c = 1$ ①

② Mostly contracting

③ $\forall \mu \in E_f$ has $\lambda^{cs}(\mu) < 0$ ← satisfied by Mañé's example

① $\cap$ ② $\cap$ ③ is C^1 open

← most delicate. Gibbs u $\subset E_f$.

Moreover, (*) extends to a generic subset C^1.